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Comments on the β-deformed $\mathcal{N} = 4$ SYM Theory

Daniel Z. Freedman[⋆†1]
and Umut Gürsoy[†2]

[*]*Department of Mathematics*
[†] *Center for Theoretical Physics*
Massachusetts Institute of Technology
Cambridge MA 02139, USA

Abstract

Several calculations of 2- and 3-point correlation functions in the deformed theory are presented. The central charge in the Lunin-Maldacena gravity dual is shown to be independent of the deformation parameter. Calculations show that 2- and 3-point functions of chiral primary operators have no radiative corrections to lowest order in the interactions. Correlators of the operator $\mathrm{tr}(Z_1 Z_2)$, which has not previously been identified as chiral primary, also have vanishing lowest order corrections.

[1]dzf@math.mit.edu
[2]umut@mit.edu

1 Introduction

In the β-deformed $\mathcal{N}=4$ SYM theory, the 3 $\mathcal{N}=1$ chiral superfields Φ_i have interaction superpotential

$$W = h\mathrm{Tr}\,(q\Phi_1\Phi_2\Phi_3 - \frac{1}{q}\Phi_1\Phi_3\Phi_2), \tag{1.1}$$

with complex parameters h and q, the latter usually written as $q = e^{i\pi\beta}$. The undeformed $\mathcal{N}=4$ theory is obtained for $q=\pm 1$ and $|h| = 2g$, where g is the gauge coupling. The deformed theory has $\mathcal{N}=1$ SUSY with $U(1)_R$ and $U(1)\times U(1)$ flavor symmetries. It was pointed out in [1] that renormalization group beta functions vanish if one condition on the constants q, h is satisfied, giving an exactly marginal deformation of $\mathcal{N}=4$ SYM with $\mathcal{N}=1$ SUSY. New attention has been focused on this deformed theory, for gauge group $SU(N)$, after its gravity dual was found by Lunin and Maldacena[3] [2]. It is a solution of Type IIB supergravity with product metric $AdS_5\times\tilde{S}_5$, where $\tilde{S}_5$ is a deformed 5-sphere. The solution has many interesting features. Further developments have appeared in the subsequent recent literature [4]-[15].

In this note, we present several results most of which concern the properties of 2- and 3-point functions of $SU(2,2|1)$ chiral primary operators in the deformed theory in the weak coupling limit, thus exploring the perturbative dynamics of the exactly marginal deformation. For this purpose we combine the methods of [16] and Appendix D of [17]. The major conclusion is that the order $\lambda = g^2N$ radiative corrections to these correlators vanish, which is evidence that they are "protected", that is given exactly by their free field values. In the undeformed $\mathcal{N}=4$ theory, this "protected" property was first observed in the strong coupling limit through the $AdS_5\times S_5$ gravity dual [18]. It may be possible to carry out a similar analysis for the Lunin-Maldacena solution on the deformed $\tilde{S}_5$, although the problem of Kaluza-Klein decomposition of the coupled fluctuations is quite difficult. The spin chain method of calculation of anomalous dimensions, initiated in [19] for the $\mathcal{N}=4$ theory, has also been applied to the β-deformation in [6], and information on 3-point functions may also be available through this method [20]. Spin chain techniques are elegant and useful, but the simple weak coupling methods used herein lead to a clear and worthwhile picture of the correlators of chiral primaries.

[3]Earlier partial results were obtained in [3]

2 The Central Charge from the Gravity Dual

If the Lunin-Maldacena solution is indeed the dual of the β-deformed $\mathcal{N} = 4$ theory, then correlation functions computed from gravity by techniques developed early in the study of the AdS/CFT correspondence [22, 23, 24, 25] should agree with field theory correlators in the limit $N \to \infty$ and $\lambda = g^2 N >> 1$. Operators in the gauge theory are dual to fluctuations of IIB gravity theory fields about the L-M background solution. In most cases therefore, analysis of the fluctuations is required before correlation functions can be studied, and this is difficult as stated above.

One important case in which detailed fluctuation analysis is not required is the 2-point function $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$ which determines the central charge c of the CFT. An exactly marginal deformation should not change the central charge from its value $c = N^2/4$ in the undeformed $\mathcal{N} = 4$ theory. The gravity calculation of this correlator requires only the metric fluctuations in the AdS_5 part of the metric. We are thus interested in reducing the gravitational part of the IIB action on the $\tilde{S}_5$ internal space, so we write

$$\begin{aligned} S_{10} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{G_{10}}\, [R_{10} + \ldots] & (2.2) \\ &= \frac{A}{2\kappa_{10}^2} \int d^5x \sqrt{g_5} \left[R_5 + 12/L^2 + \ldots \right]. & (2.3) \end{aligned}$$

For a direct product metric, the factor A would be $Vol(\tilde{S}_5)$, as first shown in [21] and further applied in [26]. Things are slightly more complicated in the L-M case because the Einstein frame metric is a warped product given by

$$ds_E^2 = L^2 G^{-1/4} [ds^2_{AdS_5} + \sum_i (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + |\tilde{\gamma}|^2 L^4 G \mu_1^2 \mu_2^2 \mu_3^2 (\sum_i d\phi_i)^2], \quad (2.4)$$

$$\begin{aligned} G^{-1} &= 1 + |\tilde{\gamma}|^2 L^4 (\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_1^2\mu_3^2), & (2.5) \\ L^4 &= 4\pi N. & (2.6) \end{aligned}$$

The $\tilde{S}_5$ metric is parameterized by 3 angles ϕ_i and 3 positive coordinates μ_i which are constrained to satisfy $\sum_i \mu_i^2 = 1$. The complex parameter $\tilde{\gamma}$ is the parameter in the L-M solution which describes a general β-deformation with $\beta = \gamma + i\sigma$. We are using (3.24) of [2]. Denoting the $\tilde{S}_5$ metric in (2.4) by $ds^2 = h_{ij} dy^i dy^j = \sum_i (d\mu_i^2 + \ldots)$, we see that the factor A in (2.2) is given by

$$A = \int dy^1 \ldots dy^6 \sqrt{h} G^{-5/4} G^{1/4} \delta(\sum_i \mu_i^2 - 1). \quad (2.7)$$

It is now straightforward to calculate $h = \det h_{ij} = G^2 \mu_1^2 \mu_2^2 \mu_3^2$ and to observe that the *integrand* in (2.7) is independent of the deformation parameter $\tilde{\gamma}$. This is sufficient to show that the deformation, as described by the L-M solution, does not change the central charge c. The central charge in the L-M solution was also studied in [14].

Actually there is a general argument that n-point functions of $T_{\mu\nu}$ are unchanged by the deformation. Recall that the L-M solution is generated by an $SL(2,R)$ transformation which changes the modular parameter of a 2-torus in the 10 dimensional spacetime, but leaves the Einstein frame metric of the complementary 8 dimensions invariant. A $U(1) \times U(1)$ group, dual to the flavor group in the field theory, acts on the torus. Bulk fields which are invariant under $U(1) \times U(1)$, such as the 5-dimensional metric dual to the stress tensor, give boundary correlators which are unaffected by the $SL(2,R)$ transformation which produces the deformation. [4]

3 Two-point correlation functions at weak coupling.

The main purpose of this section is to study the lowest order radiative corrections to 2-point functions of various scalar operators in the deformed theory. Among other things, we will confirm the absence of anomalous dimensions for the chiral primary operators identified in [28, 29, 2], and we will find that an operator not in this list has vanishing anomalous dimension in lowest order. The techniques we develop here will be applied to 3-point correlators in the next section. It is important that we use the gauge group SU(N), since the deformed U(N) theory is not conformal.

3.1 Preliminaries

In a general $\mathcal{N} = 1$ SUSY gauge theory with chiral superfields Φ^A and general cubic superpotential

$$W = \frac{1}{6} Y_{ABC} \Phi^A \Phi^B \Phi^C, \tag{3.8}$$

[4] We are grateful to Juan Maldacena who supplied this argument.

the β-function for the couplings Y_{ABC} is determined by the anomalous dimension matrix of the elementary Φ^A via

$$\beta_{ABC} = Y_{ABD}\gamma^D_C + Y_{ADC}\gamma^D_B + Y_{DBC}\gamma^D_A. \tag{3.9}$$

To 1-loop order the anomalous dimension matrix is [30]

$$16\pi^2\gamma^A_B = \frac{1}{2}Y^{BCD}\bar{Y}_{ACD} - 2g^2C(R)^A_B, \qquad C(R)^A_B = (T^aT^a)^A_B, \tag{3.10}$$

It may also be seen from (6b) of [30] that if $\gamma^{(1)} = 0$ and $Q = T(R) - 3C(G) = 0$, then $\gamma^{(2)}$ also vanishes. Then the gauge coupling β-function β_{NSVZ} also vanishes through 3-loop order.

In the β-deformed $\mathcal{N} = 4$ theory, $\Phi^A \to \Phi^a_i$, while $C(R) = N\delta^{ab}\delta_{ij}$, $C(G) = N$, and $T(R) = 3N$, so $Q = 0$. Color symmetry and the $Z_3 \otimes Z_3$ symmetry, see [3], of the superpotential (1.1) require that γ is diagonal, i.e. $\gamma^A_B \to \gamma\delta^{ab}\delta_{ij}$. One readily obtains from (3.10)

$$\gamma^{(1)} = \left\{\frac{1}{4}|h|^2\left(N(|q|^2 + \frac{1}{|q|^2}) - \frac{2}{N}|q - \frac{1}{q}|^2\right) - 2g^2N\right\}. \tag{3.11}$$

The condition $\gamma^{(1)} = 0$ ensures that the deformed $SU(N)$ theory is conformal invariant through 2-loop order. This condition may be modified by higher loop corrections, but the diagonal property ensures that there is always a fixed point set of real codimension 1 in the space of couplings h, q, g^2. The fixed line of $\mathcal{N} = 4$ SYM theory is attained when $q = \pm 1$ and $|h|^2 = 4g^2$.

For gauge group $U(N)$, the situation is somewhat different. The color singlet fields Φ^0_i have no D-type interactions, but they do couple through the deformed superpotential

$$W_{U(N)} = \frac{1}{\sqrt{N}}h\Phi^0_1(q - \frac{1}{q})\mathrm{tr}(\Phi_2\Phi_3) + cyclic + W_{SU(N)}. \tag{3.12}$$

Singlet and non-singlet fields have different anomalous dimensions, namely

$$16\pi^2\gamma_{sing} = \frac{N}{4}|h|^2|q - \frac{1}{q}|^2 \tag{3.13}$$

$$16\pi^2\gamma_{non-sing} = \frac{N}{4}|h|^2(|q|^2 + \frac{1}{|q|^2}) - 2g^2N. \tag{3.14}$$

The marginal deformation is lost, but singlet couplings flow to zero at long distance. The IR theory is conformal with the same Lagrangian description as the $SU(N)$ theory.

We are interested in the 2-point (and later 3-point) correlators of single trace operators of the general form $\mathrm{tr}(Z_1^j Z_2^k Z_3^l)$ where Z_i is the lowest component of the superfield Φ_i. We now make two observations which greatly simplify the required calculations. We refer to the component Lagrangian which is given in the Appendix, see (7.36). The first observation concerns the relation of self-energy corrections to internal scalar lines in the undeformed and deformed theories. These corrections include a spinor loop with F-term vertices from the Yukawa interaction in (7.36). The amplitude is proportional to the term involving $|h|^2$ in $\gamma^{(1)}$ in (3.11) and thus appears to depend on the deformation parameter q. However we must work on the fixed point locus where $\gamma^{(1)} = 0$, so the spinor loop is really proportional to g^2 and independent of q.

This fact allows us to apply the arguments of [16] and Appendix B of [17] that order g^2N radiative corrections due to D-term interactions cancel with self-energy insertions in 2- and 3-point functions. The reason is that D-term effects do not depend on the flavor quantum number of the fields Z_i. So their contribution to correlators of $\mathrm{tr}Z_1^j Z_2^k Z_3^l$ is the same as for the the single flavor operators $\mathrm{tr}Z_1^{j+k+l}$. The lowest order radiative corrections to these correlators, which have no F-term contributions (other than self-energy) then vanish by the combinatoric arguments of [16]. These arguments are valid to lowest order in g^2N and all orders in $1/N$. They imply that we need only consider contributions from the quartic F-term interaction in (7.36).

Since we will be primarily interested in chiral primary operators, we now describe the chiral ring of the β-deformed theory . To classify these operators we define three separate $U(1)$ flavor groups such that the field Z_i carries unit charge under the ith $U(1)$ and the other fields are uncharged. For general β the single trace operators of the chiral ring have the following charge assignments,

$$(J,0,0), \quad (0,J,0), \quad (0,0,J), \quad (J,J,J). \tag{3.15}$$

Special chiral operators (dual to strings which wind contractible cycles in $\tilde{S}_5$ [2]) exist for the special values $\beta = m/n$ where m and n are mutually prime integers. They carry the $U(1)$ charges

$$(n_1,n_2,n_3), \qquad n_1 = n_2 = n_3 \pmod{n}. \tag{3.16}$$

As we will see from explicit calculations in the case (J, J, J), the actual operator with vanishing anomalous dimension is not $\mathrm{tr}(Z_1^J Z_2^J Z_3^J)$ but rather a specific sum over permutations of the Z_i fields involved. The same is true for the special cases (n_1, n_2, n_3).

3.2 Correlators of $\mathrm{tr}(Z_i^J) = \mathcal{O}_i^J$.

Applying the arguments above, it is obvious that the 2-point functions $\langle \mathcal{O}_i^J \; \bar{\mathcal{O}}_i^J \rangle$ (no sum on i) have no lowest order radiative corrections, since the quartic F-term interaction does not contribute. In fact, to lowest order, *all* correlators of $\mathcal{O}_i^J$ agree with those in the undeformed theory. Thus all 3-point functions and all extremal n-point functions [31], i.e.

$$\langle \mathcal{O}_i^{J_1} \; \ldots \mathcal{O}_i^{J_k} \; \bar{\mathcal{O}}_i^{J_1+\ldots+J_k} \rangle \tag{3.17}$$

are protected at lowest order. However, non-extremal correctors such as

$$\langle \mathcal{O}_i^{J_1} \mathcal{O}_i^{J_2} \; \bar{\mathcal{O}}_i^{J_3} \bar{\mathcal{O}}_i^{J_4} \rangle \tag{3.18}$$

are not protected, since they have non-vanishing D-term radiative corrections.

3.3 The operators tr$(Z_i^J Z_j)$ for $i \neq j$ and $J > 1$

In the $\mathcal{N} = 4$ theory these operators have protected 2- and 3-point correlators because they are obtained by applying an SU(3) lowering operator to $\mathrm{tr}Z_i^{J+1}$ which is clearly part of the symmetrized traceless $\mathrm{tr}X^k$ in the common $SO(6)$ designation. Since the SU(3) symmetry is broken to $U(1) \otimes U(1)$ by the deformation we would not expect that these operators remain chiral primary.

We now develop the "effective operator" method, see Appendix D of [17], which we use in most of our calculations. This method allows us to obtain full results for two-point functions with only half the combinatoric labor.

To be definite we take $i = 1, j = 2$. Other cases can be trivially obtained from this one. Radiative corrections to the 2-point correlator $\langle \mathrm{tr}Z_1^J Z_2(x) \mathrm{tr}\bar{Z}_2 \bar{Z}_1^J(y) \rangle$ are obtained in two stages. In the first step we calculate the effective operator obtained from the Wick contractions of $Z(x)$ and $\bar{Z}(z)$ fields in one factor of the F-term vertex in (7.36),

$$\mathrm{tr}(Z_1^J Z_2) \mathrm{tr}\left(T^a (\bar{q} \bar{Z}_2 \bar{Z}_1 - \frac{1}{\bar{q}} \bar{Z}_1 \bar{Z}_2) \right). \tag{3.19}$$

We ignore propagator factors temporarily, but keep track of the q dependence. To simplify the calculations we keep only leading terms in N from the splitting/joining rules (7.38) and we drop Wick contractions corresponding to nonplanar diagrams. It is quite straightforward to obtain the effective operator

$$\mathcal{O} = (\bar{q} - \frac{1}{\bar{q}})N\mathrm{tr}(Z_1^{J-1}a) \tag{3.20}$$

In the second step we contract with the trace factor in the conjugate operator, i.e.

$$(Z_1^{J-1}a)(a\bar{Z}_1^{J-1}) = (Z_1^{J-1}\bar{Z}_1^{J-1}) = N^J \tag{3.21}$$

Finally we put things together, starting with the free-field term containing $J+1$ propagators and a factor N^{Z+1} from processing the traces. We then add the above result for the interaction with propagators restored and the regulated integral (7.37) used. The correlator is then

$$\langle \mathrm{tr}Z_1^J Z_2(x)\mathrm{tr}\bar{Z}_2\bar{Z}_1^J(y)\rangle = \frac{N^{J+1}}{(4\pi^2)^{J+1}}\frac{1}{(x-y)^{2(J+1)}}[1-\gamma\ln(M^2(x-y)^2)] \tag{3.22}$$

with anomalous dimension $\gamma = |h|^2N|q-\frac{1}{q}|^2/8\pi^2$. Note that γ vanishes when $q\to\pm 1$ which is the limit of the undeformed $\mathcal{N}=4$ theory.

The leading N approximation is a major simplification for $J>1$ and it is appropriate for comparison with gravity results using AdS/CFT. Of course it gives an incomplete answer in the field theory.

3.4 The special case tr(Z_iZ_j) for $i\neq j$

In this case it is easy to include terms of all orders in N. The effective operator is

$$\begin{aligned}
\mathcal{O} &= \mathrm{tr}(Z_iZ_j)\mathrm{tr}\left(T^a(\bar{q}\bar{Z}_j\bar{Z}_i - \frac{1}{\bar{q}}\bar{Z}_i\bar{Z}_j)\right) && (3.23)\\
&= (\bar{q}-1/\bar{q})[\frac{N^2-1}{N}(1-\frac{1}{N})(T^a)] && (3.24)
\end{aligned}$$

which vanishes in $SU(N)$, indicating that to lowest order the operator behaves as a chiral primary.

Of course, this could be an accident of lowest order perturbation theory, and higher loop corrections should be studied. At 3-loop order there are

many contributing Feynman diagrams including both quartic and Yukawa interactions from (7.36), so this appears to be a difficult problem. Supergraph methods, as used in [32], may be advantageous.[5]

It is curious to repeat the lowest order calculation for gauge group $U(N)$, the effective operator becomes

$$\mathcal{O} = (q - \frac{1}{q})N^{\frac{3}{2}}\delta^{a0}. \tag{3.25}$$

The 2-point function then takes the form (3.22) with $\gamma = |h|^2 N|q - \frac{1}{q}|^2/8\pi^2$ and $J = 1$. So the operator acquires anomalous dimension in this case.

3.5 $(1,1,1)$ operator

In this sector we expect that there is one linear combination $\mathrm{tr}(Z_1Z_2Z_3) + \alpha\,\mathrm{tr}(Z_1Z_3Z_2)$ which has vanishing anomalous dimension. We apply the splitting joining rules in (7.38) to work out the effective operators

$$\begin{aligned}
\mathcal{O}_{123} &= \mathrm{tr}(Z_1Z_2Z_3)\mathrm{tr}\left(T^a(\bar{q}\bar{Z}_2\bar{Z}_1 - \frac{1}{\bar{q}}\bar{Z}_1\bar{Z}_2)\right) \\
&= [N\bar{q} - \frac{2}{N}(\bar{q} - \frac{1}{\bar{q}})]\mathrm{tr}(Z_3T^a) \\
\mathcal{O}_{132} &= \mathrm{tr}(Z_1Z_3Z_2)\mathrm{tr}\left(T^a(\bar{q}\bar{Z}_2\bar{Z}_1 - \frac{1}{\bar{q}}\bar{Z}_1\bar{Z}_2)\right) \\
&= -[\frac{N}{\bar{q}} - \frac{2}{N}(\bar{q} - \frac{1}{\bar{q}})]\mathrm{tr}(Z_3T^a)
\end{aligned} \tag{3.26}$$

One must add the similar contributions from the other two cyclic permutations in the F-term Lagrangian. These give contributions which differ from (3.26) only by the replacements $\mathrm{tr}(Z_3T^a) \to \mathrm{tr}(Z_1T^a)$ and $\mathrm{tr}(Z_3T^a) \to \mathrm{tr}(Z_2T^a)$.

The 1-loop anomalous dimension is obtained by contracting the effective operators for the linear combinations $\mathrm{tr}(Z_1Z_2Z_3) + \alpha\,\mathrm{tr}(Z_1Z_3Z_2)$ with their conjugates. It is clear that the anomalous dimension vanishes if and only

[5] In [33] it has been shown that the anomalous dimension of $\mathrm{tr}(Z_iZ_j)$ vanishes to 3-loop order.

if the operator itself vanishes. This fixes the complex coefficient α. For arbitrary q and N one obtains the following BPS operator:

$$\mathcal{O}^{(1,1,1)} = \mathrm{tr}(Z_1 Z_2 Z_3) + \frac{(N^2-2)\bar{q}^2 + 2}{N^2 - 2 + 2\bar{q}^2}\ \mathrm{tr}(Z_1 Z_3 Z_2). \qquad (3.27)$$

To leading order in N, the coefficient $\alpha \to \bar{q}^2$.

3.6 $(2,2,0)$ operator

It is straightforward to use the effective operator method to investigate the linear combination $\mathrm{tr}(Z_1Z_1Z_2Z_2) + c\,\mathrm{tr}(Z_1Z_2Z_1Z_2)$. To leading order in N, the effective operator is

$$\mathcal{O} = N[(2c\bar{q} - \frac{1}{\bar{q}})\mathrm{tr}(Z_1 Z_2 T^a) + (\bar{q} - \frac{2c}{\bar{q}})\mathrm{tr}(Z_2 Z_1 T^a)]. \qquad (3.28)$$

The anomalous dimension vanishes to 1-loop order only if $q^4 = 1$, which is what we expect from the classification of chiral primary operators. There are two cases to consider:
i) $q = \pm 1$ with $c = \frac{1}{2}$. This means that we have the undeformed $\mathcal{N} = 4$ SYM theory, and the operator is the second $SU(3)$ descendent of the primary $\mathrm{tr}(Z_1^4)$.
ii) $q = \pm i$ with $c = -\frac{1}{2}$. This is the expected chiral primary chiral operator of type $(2,2,0)$ which occurs for the special value $\beta = 1/2$. With $1/N$ corrections included, this operator becomes

$$\mathcal{O}^{(2,2,0)} = \mathrm{tr}(Z_1Z_1Z_2Z_2) - \frac{1}{2}\left(\frac{N^2-8}{N^2-4}\right)\ \mathrm{tr}(Z_1Z_2Z_1Z_2). \qquad (3.29)$$

3.7 BPS operators at large N

So far, we constructed some special members of the chiral primary operators. Our results above are valid for all values of N. It becomes cumbersome determine the chiral primaries that are composed of more canonical fields by the method of vanishing effective operator. However, one can easily work out the general form of these operators when N is large, as below.

The (J_1, J_2, J_3) operator is given as,

$$\mathcal{O} = \sum_\pi c_\pi\ \mathrm{tr}(\pi \cdot Z_1^{J_1} Z_2^{J_2} Z_3^{J_3}) \qquad (3.30)$$

where π is the sum over all distinct permutations modulo cyclicity of the trace[6]. Determination of the the complex coefficients c_π, turns out to be easy in the large N limit:

$$c_\pi = \frac{\bar{q}^{2k_\pi}}{s_\pi}, \tag{3.31}$$

where k is a positive integer and s is a symmetry factor of the permutation.

To specify k_π and s_π we express the permutation π in terms of the elementary exchanges (12), (23) and (31). We assign $c_1 = 1$ for the identity permutation. Then k_π is obtained by introducing a factor $\bar{q}^2$ for each exchange. For example,

$$223311 \to \bar{q}^4 212313.$$

The symmetry factor is the number of repeated arrays in the permutation: *e.g.* for 213213 $s = 2$, for 212121 $s = 3$. Division by the symmetry factor is required for the cancellation among contributions to the effective operators.

Let us illustrate the basic mechanism for the cancelation of the one-loop radiative corrections to (3.30) with the special $(n, n, 0)$ operators. They are chiral primary when $\beta = m/n$. These operators are conjectured to be dual to strings that rotate along the two contractible cycles in the deformed S^5 with momenta $n_1 = n_2 = n$ and winding numbers $w_1 = -m$, $w_2 = m$ [2]. Consider the F-term interactions of the following array of fields.

$$(Z_2 \cdots Z_2 Z_1 \cdots Z_1) \tag{3.32}$$

The quartic interaction that involves the pair of fields in the middle of this array gives a contribution to the effective operator that is proportional to

$$-\frac{1}{\bar{q}}(Z_2 \cdots T^a \cdots Z_1)$$

in the limit $N \to \infty$. In case of the special BPS operator this contribution cancels out the contribution that comes from a term in (3.30) that is obtained from (3.32) by shifting the position of Z_2 one step right. This contribution is proportional to,

$$\bar{q}\left(\frac{1}{\bar{q}}\right)^2 (Z_2 \cdots T^a \cdots Z_1)$$

in the limit $N \to \infty$ so it cancels out the first contribution above. Similarly, one can see that all of the F-term contributions are paired up in a manner

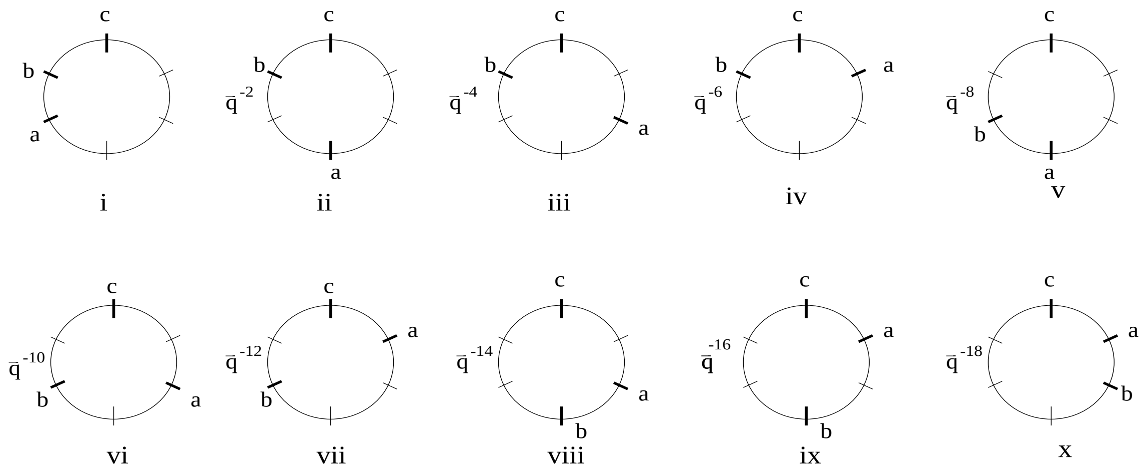

Figure 1: Terms in the $(3,3,0)$ operator shown together with their coefficients. It is easy to see that this expansion is the same as (3.33) when the condition $q^6 = 1$ is used. Bold marks denote the positions of Z_2 fields, and the smaller marks denote Z_1's. F-terms interactions of a with its neighbors is cancelled out among the group of terms i-ii-iii-iv, v-vi-vii and viii-ix seperately. Similarly the interactions of b is cancelled out among the groups ii-v, iii-vi-viii and iv-vii-ix-x. The interactions of c is cancelled by the same mechanism among the groups i-v-viii-x, ii-vi-ix and iii-vii *provided that* $q^6 = 1$.

similarly and cancel out provided that $q^{2n} = 1$.

This last requirement can be seen as follows. There are n Z_2 fields in the $(n,n,0)$ operator that pair up with their neighbor Z_1 fields to form an F-quartic vertex. Let us label these fields by a, b, $c, \ldots$ as in the fig. 1. As can easily be seen from the figure the interactions of the first $n-1$ Z_2 fields cancel out by the above mechanism. However in order to cancel the interactions of the nth field by the same mechanism one needs to impose the cyclicity condition $q^{2n} = 1$. Note that this requirement coincides with the definition of the chiral ring.

We list several examples of the prescription (3.32). The case of $n = 2$ is already discussed at the end of the previous section where it was simple enough to determine the operator for all orders in N. In the next case $n = 3$

[6] It may be interesting to derive this definition of the chiral primaries from a $q-algebra$ method as in [34].

one has the following two special operators (for large N),

$$\begin{aligned}\mathcal{O}^{(3,3,0)} &= \operatorname{tr}(Z_2Z_2Z_2Z_1Z_1Z_1) + q^2 \operatorname{tr}(Z_2Z_2Z_1Z_2Z_1Z_1) + \\ &\quad + q^4 \operatorname{tr}(Z_2Z_2Z_1Z_1Z_2Z_1) + \frac{1}{3}\operatorname{tr}(Z_2Z_1Z_2Z_1Z_2Z_1)\end{aligned} \quad (3.33)$$

where $q = e^{2i\pi/3}$ and $q = e^{4i\pi/3}$. These two operators are related to each other by the relabeling of the fields $Z_1 \leftrightarrow Z_2$.

Finally, for $n = 4$, q-variation method results in the following special BPS operators (for large N),

$$\begin{aligned}\mathcal{O}^{(4,4,0)} &= \operatorname{tr}(Z_2Z_2Z_2Z_2Z_1Z_1Z_1Z_1) + q^2 \operatorname{tr}(Z_2Z_2Z_2Z_1Z_2Z_1Z_1Z_1) \\ &\quad + q^4 \operatorname{tr}(Z_2Z_2Z_2Z_1Z_1Z_2Z_1Z_1) + q^6 \operatorname{tr}(Z_2Z_2Z_2Z_1Z_1Z_1Z_2Z_1) \\ &\quad + q^4 \operatorname{tr}(Z_2Z_2Z_1Z_2Z_2Z_1Z_1Z_1) + q^6 \operatorname{tr}(Z_2Z_2Z_1Z_2Z_1Z_2Z_1Z_1) \\ &\quad + \operatorname{tr}(Z_2Z_2Z_1Z_2Z_1Z_1Z_2Z_1) + \frac{1}{2}\operatorname{tr}(Z_2Z_2Z_1Z_1Z_2Z_2Z_1Z_1) \\ &\quad + q^6 \operatorname{tr}(Z_2Z_2Z_1Z_1Z_2Z_1Z_2Z_1) + \frac{q^4}{4}\operatorname{tr}(Z_2Z_1Z_2Z_1Z_2Z_1Z_2Z_1)\end{aligned}$$

where the independant choices for q are $e^{i\pi/4}$ and $e^{i\pi/2}$ and $e^{i3\pi/4}$.

It is straightforward to check that the F-term radiative corrections to the two-point functions of these operators vanish as $N \to \infty$. However we expect to obtain modifications for finite N.

It is straightforward but somewhat more complicated to show that the prescription (3.30) determines an operator with vanishing anomalous dimension for all values of J_1, J_2, J_3 which correspond to chiral primaries.

4 Three point functions of chiral primary operators

In this section we consider three-point correlators of the chiral primary operators listed in (3.15, 3.16) and their complex conjugates. We note that all three-point functions are extremal and can be schematically written as $\langle \bar{\mathcal{O}}\mathcal{O}_1\mathcal{O}_2 \rangle$ where $\mathcal{O}_1$ and $\mathcal{O}_2$ contain only $Z_{1,2,3}$ and $\bar{\mathcal{O}}$ contains only conjugate fields.

As discussed in Sec 3.1 above, the combinatoric arguments of [16] imply that the D-term interactions cancel provided that one works on the fixed point locus where $\gamma^{(1)} = 0$.

We now argue that lowest order F-term interactions also vanish because the effective operator method gives the same operator which occurred in the computation of two-point functions, and that operator thus vanishes. The one-loop radiative correction is given by Wick contractions of the three point function with the F-term quartic vertex:

$$\sum_{i,a} \langle \overline{\mathcal{O}} \frac{\partial W}{\partial Z_i^a} \frac{\partial \bar{W}}{\partial \bar{Z}_i^a} \mathcal{O}_1^{J_1}(y) \mathcal{O}_2^{J_2} \rangle. \tag{4.34}$$

Wick contractions of $\overline{\mathcal{O}} \frac{\partial W}{\partial Z_i^a}$ give the same effective operator $\overline{\mathcal{O}}_i^a$ which occurs in the computation of the F-term corrections to the 2-point function of $\overline{\mathcal{O}}$. These corrections vanish if $\overline{\mathcal{O}}$ is chiral primary. Therefore the one-loop radiative correction to the three point function is absent.

These arguments generalize immediately to extremal n-point functions of all chiral primaries.

5 Conclusions

We have shown that lowest order calculations by the effective operator method confirm the assignment of chiral primary operators of [29] and suggest that their two- and three-point correlation functions (and extremal correlators) have vanishing radiative corrections. It would be interesting to confirm this suggestion in further study and to ascertain the properties of $\mathrm{tr}(Z_i Z_j)$, with $i \neq j$, which we have shown to be protected to lowest order although it was not previously recognized as a chiral primary. There are far fewer chiral primary operators in the β-deformed theory than in its undeformed $\mathcal{N} = 4$ parent. But it is striking that their correlators appear to enjoy the same properties that have been established in the latter case.

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7 Appendix

7.1 Conventions

We use the following conventions:

$$\mathrm{tr}(T^a T^b) = \delta^{ab}, \qquad [T^a, T^b] = i\sqrt{2} f^{abc} T^c, \tag{7.35}$$

With these conventions, the action of the deformed theory can be written in the $\mathcal{N}=1$ component notation as follows:

$$\begin{aligned}
\mathcal{L} \;=\; & \mathrm{tr}\Big[\frac{1}{4}{F_{\mu\nu}}^2 + \frac{1}{2}\bar{\lambda}\not{D}\lambda + \overline{D_\mu Z^i} D_\mu Z^i + \frac{1}{2}\bar{\psi}^i \not{D}\psi^i + g(\bar{\lambda}\bar{Z}^i L\psi^i - \bar{\psi}^i R Z^i \lambda) \\
& + \frac{1}{4} g^2 [\bar{Z}^i, Z^i]^2\Big] \\
& + \frac{h}{2}\mathrm{tr}\left(q\bar{\psi}_2 L Z_3 \psi_1 - \frac{1}{q}\bar{\psi}_1 L Z_3 \psi_2\right) + \frac{\bar{h}}{2}\mathrm{tr}\left(\bar{q}\bar{\psi}_1 R \bar{Z}_3 \psi_2 - \frac{1}{\bar{q}}\bar{\psi}_2 R \bar{Z}_3 \psi_1\right) \\
& + |h|^2 \mathrm{tr}\left(T^a(q Z_2 Z_3 - \frac{1}{q} Z_3 Z_2)\right) \mathrm{tr}\left(T^a(\bar{q}\bar{Z}_3\bar{Z}_2 - \frac{1}{\bar{q}}\bar{Z}_2\bar{Z}_3)\right) + \text{cyclic}
\end{aligned} \tag{7.36}$$

where $D_\mu Z = \partial_\mu - i\frac{g}{\sqrt{2}}[A_\mu, Z]$ and similarly for the fermions. The first two lines contain D-type interactions, and the last two lines give the F-type interactions. Note that cylic permutations of the labels 123 must be added in the F-terms. It is the quartic F-type interaction of the scalars that plays the major role in our computations. The splitting/joining rules could be used to simplify this term, but the factored form is the most convenient for the "effective operator" method used in most of our calculations.

The Lagrangian above is valid for Euclidean signature in which we consider Wick contractions with $e^{-S_{int}}$. The scalar propagator is $\langle Z(x)\bar{Z}(y)\rangle = 1/4\pi^2(x-y)^2$. The "bubble graph" with 2-quartic vertices leads to the spacetime integral

$$\frac{1}{(2\pi)^4}\int \frac{d^4 z}{z^4 (z-x)^4} \;=\; \frac{\ln M^2 x^2}{8\pi^2 x^4}. \tag{7.37}$$

This result is obtained by (partial) differential regularization, see [35].

Our methd of calculation is based on the splitting/joining rules for traces [36, 17]. We present these in a compact notation in which the generators of the fundamental of $SU(N)$ are replaced by their index values, i.e. $T^a \rightarrow$

$a, \;\; a = 1, \ldots, N^2 - 1$, and the trace of an arbitrary $N \times N$ matrix is denoted by $\mathrm{Tr}\,(M) \to (M)$. The following rules can then be used to evaluate traces and products thereof which involve sums over repeated indices:

$$\begin{array}{rclcrcl} (MaM'a) & = & (M)(M') - \frac{1}{N}(MM') & \qquad & (ab) & = & \delta^{ab} \\ (Ma)(aM') & = & (MM') - \frac{1}{N}(M)(M') & & (a) & = & 0 \\ aa & = & \frac{N^2-1}{N} I & & (I) & = & N \end{array} \tag{7.38}$$

To treat $U(N)$ we add the generator $T^0 = I/\sqrt{(}N)$. The relations in the first column of (7.38) are valid for $U(N)$ if the $\frac{1}{N}$ terms are dropped. The relations in the second column are valid except for the change $(a) \to \sqrt{N}\delta^{a0}$.

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